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Constrained stabilization of periodic discrete-time systems via periodic Lyapunov functions

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Abstract: This article considers the problem of constrained stabilization of periodically time-varying discrete-time systems, or shortly, periodic systems. A modification of a recent result on periodic Lyapunov functions, which are required to decrease at every period rather than at every time instant, is exploited to obtain a new stabilizing controller synthesis method for periodic systems. We demonstrate that for the relevant class of linear periodic systems subject to polytopic state and input constraints, the developed synthesis method is advantageous compared to the standard Lyapunov synthesis method. An illustrative example demonstrates the effectiveness of the proposed method.

Keywords: Periodic systems, periodic control laws, constrained control, Lyapunov methods

1. INTRODUCTION

Periodically time-varying systems, or shortly, *periodic systems*, represent an important system class for both control theory and applications. Some of the most relevant real-life problems are magnetic satellite control problems (Wisniewski, 1996; Psiaki, 2001) and the control of helicopter rotors (Arcara et al., 2000). Furthermore, as it was pointed out by (Colaneri et al., 1992; Longhi, 1994; Sågfors et al., 2000), time-invariant systems that are controlled by multirate synchronous inputs can be modeled by periodically time-varying systems as well.

The importance of periodic systems is also evident from the number of the relevant works in the literature. Existing control synthesis approaches concern output feedback schemes (De Souza and Trofino, 2000), $\mathcal{H}_2$ synthesis for the case of linear periodic systems with polytopic uncertainties (Farges et al., 2007), eigenvalue assignment (Brunovský, 1970; Varga, 2000), exploitation of controllability concepts (Longhi and Zulli, 1995), model matching (Colaneri and Kucera, 1997), and more recently, predictive control (Böhm, 2011). In the recent monograph by Bittanti and Colaneri (2009), Chapter 13, a thorough exposition of existing results on stabilization techniques is presented.

In this article, we focus on constrained synthesis techniques based on Lyapunov functions. Under the assumption of continuity of the system dynamics, Massera (1949) and Jiang and Wang (2002) proved that a periodically time-varying nonlinear system is (uniformly globally) asymptotically stable if and only if it admits a periodically time-varying Lyapunov function (LF). Analogously for the linear case, the well-known Periodic Lyapunov Lemma (PLL) (see for example (Bittanti and Colaneri, 2009)), establishes existence of a quadratic periodically time-varying LF for asymptotically stable periodic systems. More recently, in (Böhm et al., 2012), the introduction of *periodic Lyapunov functions* offered a useful relaxation for the problem of estimating the region of attraction for state constrained autonomous periodic systems; the Lyapunov function was not required to decrease at each sampling instant, as in (Jiang and Wang, 2002) or in (Bittanti and Colaneri, 2009) for the linear case, but at each period. The benefit of this approach was demonstrated for linear periodic autonomous systems under state constraints.

For the case of non–autonomous periodic systems, however, applying the relaxed periodic Lyapunov conditions of Böhm et al. (2012) is not practical, even for the linear periodic case, since the set of conditions that needs to be solved in the control synthesis approach is nonlinear and nonconvex. To tackle the problem, we first propose a reformulation of the periodic Lyapunov conditions for the general nonlinear case. Moreover, by choosing quadratic periodic Lyapunov functions, we show how the constrained synthesis problem for the linear case with linear periodic state feedback can be solved by decomposing the original nonconvex optimization problem in a finite set of linear matrix inequalities (LMIs). The effectiveness of the result is demonstrated in an illustrative example in which the new approach is compared with the stabilization technique stemming from direct application of the Periodic Lyapunov Lemma to synthesis.

The remainder of this paper is structured as follows. Notation and the description of the dynamics under study is provided in Section 2, along with the exposition of existing results on Lyapunov stability and a preliminary result. The main result concerning the constrained control synthesis for linear periodic systems with linear periodic state feedback is presented in Section 3. An illustrative example showing the effectiveness of the proposed synthesis technique is presented in Section 4, while conclusions are drawn in Section 5.

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2. PROBLEM SETTING AND PRELIMINARY RESULT

Let $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$ and $\mathbb{Z}_+$ denote the field of real numbers, the set of non-negative reals, the set of integer numbers and the set of non-negative integers, respectively. For every $c \in \mathbb{R}$ and $\Pi \subseteq \mathbb{R}$ we define $\Pi_{\geq c} := \{x \in \Pi \mid x \geq c\}$, and similarly $\Pi_{\leq c}$, $\mathbb{R}_\Pi := \Pi$ and $\mathbb{Z}_\Pi := \mathbb{Z} \cap \Pi$. For $N \in \mathbb{Z}_{\geq 1}$, $\Pi^N := \Pi \times \ldots \times \Pi$. For a vector $x \in \mathbb{R}^n$, $[x]_i$ denotes the i-th element of x and $\|x\|$ denotes its 2-norm, i.e., $\|x\| := \sqrt{\sum_{i=1}^n |[x]_i|^2}$. The transpose of a matrix $X \in \mathbb{R}^{n \times m}$ is denoted by $X^\top$. For a symmetric matrix $Z \in \mathbb{R}^{n \times n}$ let $Z \succ 0 (\succeq 0)$ denote that Z is positive definite (semi-definite). For a positive definite matrix $Z \in \mathbb{R}^{n \times n}$ let $\lambda_{\min(\max)}(Z)$ denote its smallest (largest) eigenvalue. Moreover, for a block symmetric matrix $Z = \left[\begin{smallmatrix} a & b^\top \\ b & c \end{smallmatrix}\right]$, where a, b, c are matrices of appropriate dimensions, the symbol $\star$ is used to denote the symmetric part, i.e., $\left[\begin{smallmatrix} a & b^\top \\ b & c \end{smallmatrix}\right] = \left[\begin{smallmatrix} a & \star \\ b & c \end{smallmatrix}\right]$. A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *belongs to class* $\mathcal{K}$ if it is continuous, strictly increasing and $\varphi(0) = 0$. A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *belongs to class* $\mathcal{K}_\infty$ if $\varphi \in \mathcal{K}$ and $\lim_{s \to \infty} \varphi(s) = \infty$. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ *belongs to class* $\mathcal{KL}$ if for each fixed $k \in \mathbb{R}_+$, $\beta(\cdot, k) \in \mathcal{K}$ and for each fixed $s \in \mathbb{R}_+$, $\beta(s, \cdot)$ is decreasing and $\lim_{k \to \infty} \beta(s, k) = 0$.

2.1 System description

Let $n, m \in \mathbb{Z}_+$ be integers and let $\mathbb{X} : \mathbb{Z}_+ \to \mathbb{R}^n$ and $\mathbb{U} : \mathbb{Z}_+ \to \mathbb{R}^m$ be maps that assign to each $k \in \mathbb{Z}_+$ a subset of $\mathbb{R}^n$ and a subset of $\mathbb{R}^m$ respectively, which contain the origin in their interior. We consider time–varying nonlinear systems of the form

$$x(k+1) = f(k, x(k), u(k)), \quad k \in \mathbb{Z}_+, \tag{1}$$

where $f : \mathbb{Z}_+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is an arbitrary nonlinear map such that $f(k, 0, 0) = 0$, for all $k \in \mathbb{Z}_+$. The vector $x(k) \in \mathbb{X}(k)$ is the system state at time $k \in \mathbb{Z}_+$ and $u(k) \in \mathbb{U}(k)$ is the system input at time $k \in \mathbb{Z}_+$. In this article, we focus on the subclass of periodically time–varying nonlinear systems which are defined as follows.

Definition 1. The system (1) is called *periodic* if there exists an $N \in \mathbb{Z}_{\geq 1}$ such that for all $k \in \mathbb{Z}_+$ it holds
(i) $\mathbb{X}(k) = \mathbb{X}(k+N)$;
(ii) $\mathbb{U}(k) = \mathbb{U}(k+N)$;
(iii) $f(k, x, u) = f(k+N, x, u)$ for all $x \in \mathbb{X}(k)$, for all $u \in \mathbb{U}(k)$.
Furthermore, the smallest such $N \in \mathbb{Z}_{\geq 1}$ is called the *period* of system (1).

We consider a periodically time–varying state feedback control law $g : \mathbb{Z}_+ \times \mathbb{R}^n \to \mathbb{R}^m$ such that $g(k, 0) = 0$, for all $k \in \mathbb{Z}_+$, $g(k, x) = g(k+N, x)$, for all $k \in \mathbb{Z}_+$, and $g(k, x(k)) \in \mathbb{U}(k)$, for all $k \in \mathbb{Z}_+$ and for all $x(k) \in \mathbb{X}(k)$. We assume, for simplicity, that the period of the control law is equal to the period of system (1). The corresponding closed–loop system is

$$x(k+1) = f(k, x(k), g(k, x(k))), \quad k \in \mathbb{Z}_+. \tag{2}$$

System (2) is periodic with period N, since $f(k+N, x, g(k+N, x)) = f(k, x, g(k, x))$.

In what follows, let $\mathbb{X}_0 := \mathbb{X}(0)$ and define $\overline{\mathbb{X}} := \bigcup_{k=0}^{N-1} \mathbb{X}(k)$. As such, all state trajectories of system (2) with $x(0) \in \mathbb{X}_0$ satisfy $x(k) \in \overline{\mathbb{X}}$, for all $k \in \mathbb{Z}_+$. Allowing for a different state-space domain for each period is beneficial not only to allow for periodically time-varying constraints, but also to accommodate periodic systems with a time-varying state dimension, see, e.g., (Sågfors et al., 2000). The classical time-invariant unconstrained state-space and input domain is recovered by setting $\mathbb{X}(k) = \mathbb{R}^n$, $\mathbb{U}(k) = \mathbb{R}^m$, for all $k \in \mathbb{Z}_+$.

We consider also the subclass of non-autonomous linear periodic systems of the form

$$x(k+1) = A(k)x(k) + B(k)u(k), \tag{3}$$

where $A(k) \in \mathbb{R}^{n \times n}$, $B(k) \in \mathbb{R}^{n \times m}$, and $A(k) = A(k+N)$, $B(k) = B(k+N)$, for all $k \in \mathbb{Z}_+$. Equivalently to the nonlinear case, choosing a linear periodic state feedback control law with period N, i.e.,

$$u(k) = g(k, x(k)) := K(k)x(k), \tag{4}$$

with $K(k) = K(k+N)$, the closed-loop system is

$$x(k+1) = (A(k) + B(k)K(k))x(k). \tag{5}$$

2.2 Stability of periodic systems

The notions of asymptotic stability, exponential stability and region of attraction for system (2) are recalled next.

Definition 2. System (2) is asymptotically stable in $\mathbb{X}_0$, or shortly, AS($\mathbb{X}_0$), if there exists a $\mathcal{KL}$-function $\beta(\cdot, \cdot)$ such that, for each $x(0) \in \mathbb{X}_0$ the corresponding state trajectory of (2) satisfies

$$\|x(k)\| \leq \beta(\|x(0)\|, k), \ \forall k \in \mathbb{Z}_+. \tag{6}$$

System (2) is called exponentially stable in $\mathbb{X}_0$, or shortly, ES($\mathbb{X}_0$), if $\beta(s, k) := \theta \mu^k s$ for some $\theta \in \mathbb{R}_{\geq 1}$, $\mu \in \mathbb{R}_{[0,1)}$.

Definition 3. Suppose that system (2) is AS($\mathbb{X}_0$). Then, the region of attraction (ROA) of (2) in $\mathbb{X}_0$ is given by the following set of initial conditions:

$$\mathcal{R}(\mathbb{X}_0) := \Big\{ \xi \in \mathbb{X}_0 \mid x(k) \in \mathbb{X}(k), \ \forall k \in \mathbb{Z}_+, x(0) = \xi, \lim_{k \to \infty} x(k) = 0 \Big\}. \tag{7}$$

Next, the notion of a *periodically positively invariant* (PPI) sequence of sets, introduced by Böhm et al. (2012), is recalled. It is worth to mention that a similar concept was also considered by Lee and Kouvaritakis (2006). Let $\{\mathbb{D}(\pi)\}_{\pi \in \mathbb{Z}_{[0,N-1]}}$ denote a sequence of sets with $\mathbb{D}(\pi) \subseteq \mathbb{X}(\pi)$ for all $\pi \in \mathbb{Z}_{[0,N-1]}$.

Definition 4. The sequence $\{\mathbb{D}(\pi)\}_{\pi \in \mathbb{Z}_{[0,N-1]}}$ is called periodically positively invariant for system (2) if for each $\pi \in \mathbb{Z}_{[0,N-1]}$, each $k \in \{iN + \pi\}_{i \in \mathbb{Z}_+}$ and $x(k) \in \mathbb{D}(\pi)$, it holds that $x(k+N) \in \mathbb{D}(\pi)$ and $x(k+j) \in \mathbb{X}(k+j)$, for all $j \in \mathbb{Z}_{[1,N-1]}$.

Note that this definition implies also satisfaction of the constraints $x(k) \in \mathbb{X}(k)$ and $u(k) \in \mathbb{U}(k)$, for all $k \in \mathbb{Z}_+$.

Assumption 5. The sequence $\{\mathbb{X}(k)\}_{k \in \mathbb{Z}_{[0,N-1]}}$ is periodically positively invariant for system (2).

Necessary and sufficient conditions for stability have been formulated for the nonlinear periodic case, see for example (Jiang and Wang, 2002). Therein, it was established that a periodically time-varying nonlinear system is (uniformly globally) asymptotically stable if and only if it admits a periodically time-varying Lyapunov function (LF).

Theorem 6. (Jiang and Wang, 2002). Let $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\rho(k) \in \mathbb{R}_{[0,1)}$, $k \in \mathbb{Z}$, and let $x(\cdot)$ be a solution to (2) with $x(0) := \xi \in \mathbb{X}(0)$. Let $V : \mathbb{Z}_+ \times \overline{\mathbb{X}} \to \mathbb{R}_+$ be a function, such that

$V(k,x) = V(k+N,x)$, for all $k \in \mathbb{Z}_+$, and moreover, for all $k \in \mathbb{Z}_+$ it holds that

$$\alpha_1(\|\xi\|) \leq V(k,\xi) \leq \alpha_2(\|\xi\|), \forall \xi \in \mathbb{X}(k) \quad \text{(8a)}$$
$$V(k+1, f(k,x(k),g(k,x(k)))) \leq \rho(k)V(k,x(k)), \quad \text{(8b)}$$

for all $\xi \in \mathbb{X}(0)$. Then, system (2) is AS($\mathbb{X}_0$).

Relation (8b) is equivalent to the following condition

$$V(k+1, f(k,x(k),g(k,x(k)))) - V(k,x(k)) \leq -\alpha_3(\|x(k)\|),$$

where $\alpha_3 \in \mathcal{K}_\infty$, see (Jiang and Wang, 2002). Using (8b) allows comparison with the other relevant results in the literature.

The following relaxation regarding the Lyapunov function was proposed in (Böhm et al., 2012).

Theorem 7. (Böhm et al., 2012). Let α_1, α_2 $\bar{\alpha}_j, j \in \mathbb{Z}_{[1,N-1]}$ be $\mathcal{K}_\infty$ functions, $\eta \in \mathbb{R}_{[0,1)}$ and $x(\cdot)$ be a solution to (2) with $x(0) := \xi \in \mathbb{X}(0)$. Let $V : \mathbb{Z}_+ \times \overline{\mathbb{X}} \to \mathbb{R}_+$ be a function, such that $V(k,x) = V(k+N,x)$, for all $k \in \mathbb{Z}_+$, and moreover, for all $k \in \mathbb{Z}_+$, for all $j \in \mathbb{Z}_{[1,N-1]}$, it holds that

$$\|x(j)\| \leq \bar{\alpha}_j(\|x(j-1)\|), \forall \xi \in \mathbb{X}(0) \quad \text{(9a)}$$
$$\alpha_1(\|\xi\|) \leq V(k,\xi) \leq \alpha_2(\|\xi\|), \quad \forall \xi \in \mathbb{X}(k) \quad \text{(9b)}$$
$$V(k+N, x(k+N)) \leq \eta V(k,x(k)), \quad \forall \xi \in \mathbb{X}(0). \quad \text{(9c)}$$

Then, system (2) is AS($\mathbb{X}_0$).

For linear periodic systems under linear periodic state feedback control law (5) and quadratic periodically time–varying Lyapunov functions, Theorem 6 is equivalent to an application of the periodic Lyapunov lemma (PLL) for the closed–loop system, as formally stated next.

Lemma 8. (Bittanti and Colaneri, 2009). Consider system (5). Let $\rho(k) \in \mathbb{R}_{[0,1)}$, $k \in \mathbb{Z}_{[0,N-1]}$, and $P(k) \in \mathbb{S}^n_{++}$, $k \in \mathbb{Z}_{[0,N]}$ be positive definite matrices, with $P(N) := P(0)$, which define sets $\mathbb{E}(k) = \{x \in \mathbb{R}^n : x^\top P(k)x \leq 1\}$ such that $\mathbb{E}(k) \subseteq \mathbb{X}(k)$, for all $k \in \mathbb{Z}_{[0,N-1]}$. If the following matrix inequalities hold, for all $k \in \mathbb{Z}_{[0,N-1]}$

$$(A(k)+B(k)K(k))^\top P(k+1)(A(k)+B(k)K(k)) - \rho(k)P(k) \preceq 0, \quad \text{(10)}$$

then system (5) is ES($\mathbb{E}(0)$).

Analogously, Theorem 7 can also be applied in a straightforward manner to the the closed-loop periodic linear system (5). To this end, we first define the monodromy matrices $\Phi(k) \in \mathbb{R}^{n\times n}$, $k \in \mathbb{Z}_{[0,N-1]}$ (Bittanti and Colaneri, 2009):

$$\Phi(k) := \prod_{i=0}^{N-1} (A(k+i) + B(k+i)K(k+i)), \forall k \in \mathbb{Z}_{[0,N-1]}.$$

The following result is the equivalent of Theorem 7 for the linear case:

Lemma 9. (Böhm et al. (2012)). Consider system (5). Let $\eta \in \mathbb{R}_{[0,1)}$, and $P(k) \in \mathbb{S}^n_{++}$, $k \in \mathbb{Z}_{[0,N]}$ be positive definite matrices, with $P(N) := P(0)$, which define sets $\mathbb{E}(k) = \{x \in \mathbb{R}^n : x^\top P(k)x \leq 1\}$ such that $\mathbb{E}(k) \subseteq \mathbb{X}(k)$, for all $k \in \mathbb{Z}_{[0,N-1]}$. If the following matrix inequalities hold, for all $k \in \mathbb{Z}_{[0,N-1]}$

$$\Phi(k)^\top P(k)\Phi(k) - \eta P(k) \preceq 0, \quad \text{(11a)}$$
$$(A(k)+B(k)K(k))^\top P(k+1)(A(k)+B(k)K(k)) - P(k) \preceq 0, \quad \text{(11b)}$$

then system (5) is AS($\mathbb{E}(0)$).

Lemma 8 and Lemma 9 guarantee asymptotic stability for the closed–loop system (5). However, Lemma 9 is a strict relaxation of the result stated in Lemma 8; A feasible set of matrices $P(k)$, $k \in \mathbb{N}_{[0,N-1]}$, and periodic state feedback gains $K(k)$, $k \in \mathbb{N}_{[0,N-1]}$, that verifies (10), verifies relations (11) as well, while the converse is not true. This is a highly relevant observation for the case of constrained stabilization, where, besides finding a stabilizing linear periodic state-feedback control law, an estimation of the region of attraction is important. On the other hand, regarding the computational complexity of the induced synthesis methods, the calculations involved in Lemma 8 require the solution of LMIs, while, in stark contrast, finding a feasible solution that verifies Lemma 9 requires the solution of a non–convex problem.

Thus, our main focus is to establish a trade–off between the strict relaxation of Lemma 9 and tractability of Lemma 8 for constrained synthesis. As a first step, the following result provides conditions of asymptotic stability for general periodic discrete–time systems.

Theorem 10. Let $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, scalars $\rho(j) \in \mathbb{R}_+$, $j \in \mathbb{N}_{[0,N-1]}$, and $x(\cdot)$ be a solution to (2) with $x(0) := \xi \in \mathbb{X}(0)$. Let $V : \mathbb{Z} \times \bar{\mathbb{X}} \to \mathbb{R}_+$ be a function, such that $V(k,x) = V(k+N,x)$, for all $k \in \mathbb{Z}_+$, and moreover, for all $j \in \mathbb{Z}_{[0,N-1]}$, it holds that

$$V(j+1, x(j+1)) \leq \rho(j)V(j,x(j)), \ \forall \xi \in \mathbb{X}(0) \quad \text{(12a)}$$
$$\alpha_1(\|\xi\|) \leq V(k,\xi) \leq \alpha_2(\|\xi\|), \ \forall \xi \in \mathbb{X}(k) \quad \text{(12b)}$$
$$0 \leq \prod_{i=0}^{N-1} \rho(i) < 1. \quad \text{(12c)}$$

Then, system (2) is AS($\mathbb{X}_0$).

Proof. From (12a) and (12b), for any $j \in \mathbb{Z}_{[1,N]}$, it holds that

$$\alpha_1(\|x(j)\|) \leq V(j,x(j)) \leq \rho(j-1)V(j-1,x(j-1)) \leq \rho(j-1)\alpha_2(\|x(j-1)\|),$$

or

$$\|x(j)\| \leq \alpha_1^{-1}(\rho(j-1)\alpha_2(\|x(j-1)\|)).$$

Thus, relation (9a) of Theorem 7 is satisfied with

$$\bar{\alpha}_j(s) := \alpha_1^{-1}(\rho(j-1)\alpha_2(s)), \quad \forall j \in \mathbb{Z}_{[1,N-1]}.$$

Moreover, under Assumption 5, from (12a), for any $k \in \mathbb{Z}_+$ and for any $x(k) \in \mathbb{X}(k)$, it holds that

$$V(k+N, x(j+N)) \leq \rho(N-1)V(j+N-1, x(j+N-1)).$$

Applying the previous inequality successively, it holds that

$$V(k+N, x(j+N)) \leq \rho(N-1)\rho(N-2)V(k+N-2, x(k+N-2)) \leq \leq ... \leq \prod_{i=0}^{N-1} \rho(i)V(k,x(k)).$$

Taking into account (12c), relation (9c) of Theorem 7 is satisfied with $\eta := \prod_{i=0}^{N-1} \rho(i) \in \mathbb{R}_{[0,1)}$. Thus, by Theorem 7, system (2) is AS($\mathbb{X}_0$). ■

It will be shown in the next section that, for the linear case, this result allows for a constructive control synthesis based on convex optimization, while still providing a strict relaxation in terms of the conditions of Lemma 8.

3. CONTROLLER SYNTHESIS

We consider linear periodic systems (3) that are subject to polytopic state periodic constraints

$$\mathbb{X}(k) := \{x \in \mathbb{R}^n : c_i(k)x \leq 1, \forall (i,k) \in \mathbb{Z}_{[1,p(k)]} \times \mathbb{Z}_+\}, \tag{13}$$

where $p(k) \in \mathbb{Z}_{\geq 1}$, for all $k \in \mathbb{Z}_+$, is the number of hyperplanes that define set $\mathbb{X}(k)$, and $c_i(k+N) = c_i(k)$, for all $(i,k) \in \mathbb{Z}_{[1,p(k)]} \times \mathbb{Z}_+$. Similarly, we consider polytopic input constraints

$$\mathbb{U}(k) := \{u \in \mathbb{R}^m : d_i(k)u \leq 1, \forall (i,k) \in \mathbb{Z}_{[1,q(k)]} \times \mathbb{Z}_+\}, \tag{14}$$

where $q(k) \in \mathbb{Z}_{\geq 1}$, for all $k \in \mathbb{Z}_+$, and $d_i(k+N) = d_i(k) \in \mathbb{R}^{1\times n}$ for all $(i,k) \in \mathbb{Z}_{[1,q(k)]} \times \mathbb{Z}_+$. In the case of state and input constraints, the simultaneous computation of a state feedback control law and a region of attraction of the closed–loop system is a challenging problem, even for the case of linear time-invariant systems with a linear state–feedback control law. In what follows, a systematic method of computing a linear periodic state feedback control law and an estimation of the region of attraction via a sequence of ellipsoidal sets will be presented.

Throughout this section, Assumption 5 is not required to hold. On the contrary, finding a periodically positively invariant sequence of sets for the closed–loop system is a synthesis objective. The problem is formally stated next:

Problem 11. Given system (3), state and input constraints $\mathbb{X}(k)$ (13) and $\mathbb{U}(k)$ (14) respectively, determine a stabilizing linear periodic state–feedback control law (4) and a corresponding PPI sequence of sets $\{\mathbb{E}(k)\}_{k\in\mathbb{Z}_{[0,N-1]}}$ with respect to the closed–loop system (5).

We consider quadratic periodic Lyapunov function candidates

$$V(k,x) = x^\top P(k)x, \tag{15}$$

where $P(k) \in \mathbb{S}^n_{++}$, with $P(k+N) = P(k)$ for all $k \in \mathbb{Z}_+$. First, we recall the necessary and sufficient conditions for inclusion of an ellipsoid within a given polytope.

Lemma 12. Let $E \in \mathbb{S}^n_{++}$ and $\psi_i \in \mathbb{R}^{1\times n}$, for all $i \in \mathbb{Z}_{[1,p]}$ with $p \in \mathbb{Z}_{\geq 1}$. The ellipsoid $\mathcal{E} := \{x \in \mathbb{R}^n : x^\top E x \leq 1\}$ is contained in the polytope $\mathcal{X} := \{x \in \mathbb{R}^n : \psi_i x \leq 1,\ i \in \mathbb{Z}_{[1,p]}\}$ if and only if $\psi_i E^{-1}\psi_i^\top \leq 1,\ \forall i \in \mathbb{Z}_{[1,p]}$.

The proof of the above result can be found in Boyd et al. (1994). The conditions of Lemma 12 allow for the calculation of an ellipsoid $\mathcal{E}$ that must be contained within a specified polytope $\mathcal{X}$.

Theorem 13. Consider system (3) and constraints $\mathbb{X}(k)$ (13), $\mathbb{U}(k)$ (14). Let $\rho(k) \in \mathbb{R}_{[0,1]}$, $X(k) \in \mathbb{S}^n_{++}$, $Y(k) \in \mathbb{S}^n_{++}$, for all $k \in \mathbb{Z}_{[0,N-1]}$, where $X(N) := X(0)$, $Y(N) := Y(0)$, be a feasible solution to the following set of matrix inequalities, for all $k \in \mathbb{Z}_{[0,N-1]}$, for all $i \in \mathbb{Z}_{[1,p(k)]}$ and all $j \in \mathbb{Z}_{[1,q(k)]}$:

$$\begin{bmatrix} \rho(k)X(k) & \star \\ A(k)X(k) + B(k)Y(k) & X(k+1) \end{bmatrix} \succeq 0, \tag{16a}$$

$$0 \leq \sum_{l=0}^{N-1} \rho(l) < N, \tag{16b}$$

$$\begin{bmatrix} 1 & \star \\ X(k)c_i(k)^\top & X(k) \end{bmatrix} \succeq 0, \tag{16c}$$

$$\begin{bmatrix} 1 & \star \\ Y(k)^\top d_j(k)^\top & X(k) \end{bmatrix} \succeq 0. \tag{16d}$$

Then, the sequence $\{\mathbb{E}(k)\}_{k\in\mathbb{Z}_{[0,N-1]}}$, where $\mathbb{E}(k) := \{x \in \mathbb{R}^n : x^\top X(k)^{-1}x \leq 1\}$, $k \in \mathbb{Z}_{[0,N-1]}$, is PPI for the closed-loop system (5) with linear periodic state feedback control law $u(k,x) = Y(k)X(k)^{-1}x$, where $u(k+N,x) = u(k,x)$ for all $k \in \mathbb{Z}_+$. Moreover, system (5) is AS($\mathbb{E}(0)$).

Proof. First, we show that the sets $\mathbb{E}(k)$, $k \in \mathbb{Z}_{[0,N-1]}$, are contained in $\mathbb{X}(k)$ and $\mathbb{U}_x(k)$ for all $k \in \mathbb{Z}_{[0,N-1]}$, where

$$\begin{aligned}\mathbb{U}_x(k) := \{x \in \mathbb{R}^n : d_i(k)Y(k)X(k)^{-1}x \leq 1,\\ \forall (i,k) \in \mathbb{Z}_{[1,q(k)]} \times \mathbb{Z}_+\},\end{aligned}$$

for all $k \in \mathbb{Z}_{[0,N-1]}$. Applying the Schur complement in (16c) and exploiting the periodicity of $\mathbb{X}(k)$ we obtain

$$c_i(k)^\top X(k)c_i(k) \leq 1, \quad \forall (i,k) \in \mathbb{Z}_{[1,p(k)]} \times \mathbb{Z}_+. \tag{17}$$

From Lemma 12, inequality (17) implies $\mathbb{E}(k) \subset \mathbb{X}(k)$, for all $k \in \mathbb{Z}_+$. Equivalently, applying the Schur complement in (16d) we obtain

$$\begin{aligned} d_j(k)Y(k)X(k)^{-1}Y(k)^\top d_j(k)^\top \leq \\ (d_j(k)Y(k)X(k)^{-1})X(k)(X(k)^{-1}Y(k)^\top d_j(k)^\top) \leq 1, \end{aligned} \tag{18}$$

for all $(j,k) \in \mathbb{Z}_{[1,q(k)]} \times \mathbb{Z}_+$. From Lemma 12, inequality (18) implies $\mathbb{E}(k) \subset \mathbb{U}_x(k)$, for all $k \in \mathbb{Z}_+$.

Next, we show that $V(k,x) = x^\top X(k)^{-1}x$ is a periodic Lyapunov function that satisfies Theorem 10 for the closed–loop system (5). The matrix inequality (16a) is equivalent to

$$\begin{aligned}(A(k)X(k) + B(k)Y(k))^\top X(k+1)^{-1}(A(k)X(k)\\ +B(k)Y(k)) - \rho(k)X(k) \preceq 0.\end{aligned}$$

Pre–multiplying and post–multiplying by $X(k)^{-1}$, the previous inequality becomes

$$\begin{aligned}(A(k) + B(k)Y(k)X(k)^{-1})^\top X(k+1)^{-1}(A(k)\\ +B(k)Y(k)X(k)^{-1}) - \rho(k)X(k)^{-1} \preceq 0.\end{aligned} \tag{19}$$

Thus, condition (12a) of Theorem 10 is satisfied with $V(k,x) = x^\top X(k)^{-1}x$. Also, condition (12b) holds with

$$\alpha_1(s) = \min_{i\in\mathbb{Z}_{[0,N-1]}} \{\lambda_{\min}(X(i)^{-1})\}s^2 \tag{20}$$

$$\alpha_2(s) = \max_{i\in\mathbb{Z}_{[0,N-1]}} \{\lambda_{\max}(X(i)^{-1})\}s^2. \tag{21}$$

Lastly, since $\rho(k) \in \mathbb{R}_{[0,1]}$, from (16b) it necessarily holds that

$$0 \leq \prod_{l=0}^{N-1} \rho(l) < 1, \tag{22}$$

thus, condition (12c) of Theorem 10 is also satisfied. Thus, from (19)–(22), Theorem 10 is satisfied, system (5) is AS($\mathbb{E}(0)$) and $\{\mathbb{E}(k)\}_{k\in\mathbb{Z}_{[0,N-1]}}$ is a PPI sequence of sets w.r.t. system (5). ■

Still, conditions (16) of Theorem 13 cannot be used directly to form a tractable synthesis method that solves Problem 11, since (16a) consists of N bilinear matrix inequalities (BMIs) due to the product of the scalars $\rho(k)$ and matrices $X(k)$, $k \in \mathbb{N}_{[0,N-1]}$. Although $\rho(k)$ is a scalar for each $k \in \mathbb{N}_{[0,N-1]}$, finding a solution to the N joint BMI conditions corresponding to (16a) is challenging, since the bisection method cannot be used. A possible approach is to grid the space of $(\rho(1), ..., \rho(N)) \in \mathbb{R}^N_{[0,1)}$ and solve the resulting LMI corresponding to (16a) for each point in the grid. Still, no convergence to a feasible solution can be guaranteed.

The next result provides a tractable way for solving Problem 11. First, consider the following problem.

Problem 14. Given system (3), constraints $\mathbb{X}(k)$ (13), $\mathbb{U}(k)$ (14), and a fixed $\bar{k} \in \mathbb{Z}_{[0,N-1]}$, solve the feasibility problem

$$\min_{X(\bar{k}),Y(\bar{k}),\bar{\rho},X(k),Y(k),k\in\mathbb{Z}_{[0,N-1]}\setminus\{\bar{k}\}} 0 \tag{23}$$

subject to

$$\begin{bmatrix} X(k) & \star \\ A(k)X(k)+B(k)Y(k) & X(k+1) \end{bmatrix} \succeq 0, \tag{24a}$$

$$\begin{bmatrix} \overline{\rho}X(\overline{k}) & \star \\ A(\overline{k})X(\overline{k})+B(\overline{k})Y(\overline{k}) & X(\overline{k}+1) \end{bmatrix} \succeq 0, \tag{24b}$$

$$0 \le \overline{\rho} < 1, \tag{24c}$$

$$\begin{bmatrix} 1 & \star \\ X(l)c_i(l)^\top & X(l) \end{bmatrix} \succeq 0, \tag{24d}$$

$$\begin{bmatrix} 1 & \star \\ Y(l)^\top d_j(l)^\top & X(l) \end{bmatrix} \succeq 0, \tag{24e}$$

with $X(N) := X(0)$, for all $k \in \mathbb{Z}_{[0,N-1]} \setminus \{\overline{k}\}$, $l \in \mathbb{Z}_{[0,N-1]}$, $i \in \mathbb{Z}_{[1,p(l)]}$, $j \in \mathbb{Z}_{[1,q(l)]}$.

Lemma 15. Consider system (3), constraints $\mathbb{X}(k)$ (13) and $\mathbb{U}(k)$ (14). Then, the matrix inequalities (16) define a nonempty feasible solution set if and only if there exists an index $\bar{k}^\star \in \mathbb{Z}_{[0,N-1]}$ such that Problem 14 is feasible with $\overline{k} = \overline{k}^\star$.

Proof. Suppose Problem 14 is feasible for a $\overline{k} = \overline{k}^\star \in \mathbb{Z}_{[0,N-1]}$. Then, relations (16) are also feasible setting $\rho(k) = 1$, for all $k \in \mathbb{Z}_{[0,N-1]} \setminus \{\overline{k}^\star\}$, $\rho(\overline{k}^\star) := \overline{\rho}$, and $X(\bar{k}^\star), Y(\bar{k}^\star)$, $X(k), Y(k)$, $k \in \mathbb{Z}_{[0,N-1]}$, the solutions to Problem 14. Conversely, suppose that conditions (16) have a nonempty feasible solution set. Then, there exists at least one $\overline{k}^\star \in \mathbb{Z}_{[0,N-1]}$ such that $\rho(\overline{k}^\star) < 1$. Setting $\bar{k} := \bar{k}^\star$ and $\overline{\rho} := \rho(\overline{k}^\star)$ the corresponding matrix inequalities (24b)–(24e) in Problem 14 are satisfied. Moreover, for any $\hat{k} \in \mathbb{Z}_{[0,N-1]} \setminus \{\bar{k}^\star\}$ such that $\rho(\hat{k}) < 1$, relation (16a) implies

$$\begin{bmatrix} X(\hat{k}) & \star \\ A(\hat{k})X(\hat{k})+B(\hat{k})Y(\hat{k}) & X(\hat{k}+1) \end{bmatrix} \succeq (1-\rho(\hat{k}))X(\hat{k}) \succeq 0.$$

Thus, (24a) is also satisfied, and consequently, Problem 14 has a solution for $\overline{k} = \overline{k}^\star$. ■

Remark 16. Comparison of conditions (16a) of Theorem 13 with condition (24b) in Problem 14, reveals the significance of the previous result. In specific, Lemma 15 shows that existence of a feasible solution to the constraint set (16), which involves N BMIs, is equivalent to existence of solution in (at least) one of the N feasibility problems (23)–(24), which involve a single bilinear term, i.e. the product of the scalar $\overline{\rho}$ and the matrix $X(\overline{k})$ in (24b). Furthermore, since the single bilinear term in (24b) consists of a matrix and the constrained nonnegative scalar $\overline{\rho} \in \mathbb{R}_{[0,1)}$, solution of Problem 14 is equivalent to solving a series of LMIs via bisection, which is guaranteed to converge to a feasible solution, if a feasible solution exists.

Remark 17. In constrained synthesis problems, together with computing a stabilizing control law, it is extremely relevant to aim for a large basin of attraction $\mathbb{E}(0) \subseteq \mathcal{R}(\mathbb{X}_0)$, where $\mathbb{E}(0) = \{x \in \mathbb{R}^n : x^\top X(0)^{-1} x \le 1\}$. To this end, we exploit the tractability of problem 14 to formulate a semi–definite optimization problem that is solved for every $\bar{k} \in \mathbb{Z}_{[0,N-1]}$, maximizes the volume of $\mathbb{E}(0)$ and solves Problem 11, i.e.,

$$\min_{X(\overline{k}),Y(\overline{k}),\overline{\rho},X(k),Y(k),k\in\mathbb{Z}_{[0,N-1]}\setminus\{\overline{k}\}} -\text{trace}(X(0)) \tag{25}$$

subject to (24).

Alternative optimization criteria that describe the size of $\mathbb{E}(0)$ can be chosen as well (for more details see (Boyd et al., 1994)).

Remark 18. Quantities $\prod_{l=0}^{N-1} \rho(l)$, where $\rho(k)$, $k \in \mathbb{N}_{[0,N-1]}$, obtained from Theorem 13, and $\overline{\rho}$, obtained from Problem 14, represent the exponential decrease of the corresponding periodic Lyapunov functions at each period, and consequently the speed of convergence of the closed–loop system trajectories. Thus, an additional benefit of the proposed method is the possibility of embedding in the synthesis procedure performance specifications. To this end, in order to achieve a desired decrease $\hat{\rho} \in \mathbb{R}_{[0,1)}$ at each period for the closed–loop system, it is sufficient to replace (24d) with $0 \le \overline{\rho} \le \hat{\rho}$. Similarly, in Theorem 13, relation (16b) can be replaced by $0 \le \sum_{l=0}^{N-1} \rho(l) \le N\sqrt[N]{\hat{\rho}}$.

4. ILLUSTRATIVE EXAMPLE

We consider a linear periodic system (3) with period $N = 2$ and system matrices $A(0) = \begin{bmatrix} 0.9 & 0.9 \\ 0.3 & 0.9 \end{bmatrix}$, $A(1) = \begin{bmatrix} 1.5 & -0.4 \\ 0.3 & 0.4 \end{bmatrix}$, $B(0) = \begin{bmatrix} 1 \\ 0.1 \end{bmatrix}$, $B(1) = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$. Also, we consider time invariant periodic polytopic state and input constraints of the form (13) and (14) respectively, such that $\mathbb{X}(0) = \mathbb{X}(1) = \mathbb{X}$, $\mathbb{U}(0) = \mathbb{U}(1) = \mathbb{U}$, where $c_1 = [1 \ \ 0]$, $c_2 = [0 \ \ 0.33]$, $c_3 = -c_1$, $c_4 = -c_2$, $d_1 = -d_2 = 0.95$.

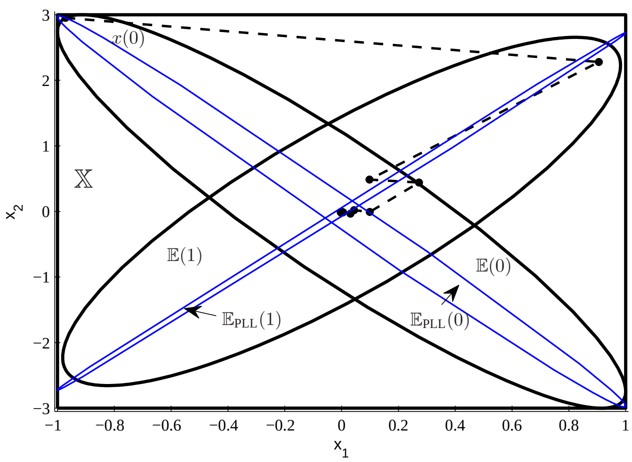

Fig. 1. The constraint set $\mathbb{X}$, the PPI sequence $\{\mathbb{E}(i)\}_{i\in\mathbb{Z}_{[0,l]}}$ computed applying the method presented (black) and the trajectory of the closed–loop system starting from $x(0) = [-0.9713 \ \ 2.9531]^\top$ (dashed black), and the PPI sequence $\{\mathbb{E}_{\mathrm{PLL}}(i)\}_{i\in\mathbb{Z}_{[0,l]}}$ computed applying the PLL (blue).

The synthesis objectives concern the calculation of a stabilizing linear periodic state feedback control law $u(k) = K(k)x(k)$, with $K(k+2) = K(k)$ for all $k \in \mathbb{Z}_+$, and an estimation of the region of attraction $\mathcal{R}(\mathbb{X})$ with a performance guarantee on the speed of convergence to the origin. For comparison, we applied both the synthesis method presented in Section 3 and the method based on the PLL Lemma (i.e. application of Lemma 8 for the closed–loop system, forming an optimization problem similar to (25), see for example (Böhm, 2011) for details). The desired performance of the closed–loop system is quantified setting $\overline{\rho} = 0.5$ in problem (25) and $\rho(k) = \sqrt{0.5}$, $k \in \mathbb{Z}_{[0,1]}$ in (10) respectively. Problem (25) was solved for two instances of $\overline{k} = 0, 1$. The chosen solution was obtained for $\overline{k} = 0$, which yielded the largest estimation of the region of attraction. For the method proposed here, the computed linear periodic feedback gains were found to be $K(0) = [-0.4207 \quad -0.4357]$, and $K(1) = [-1.3445 \quad 0.2281]$.

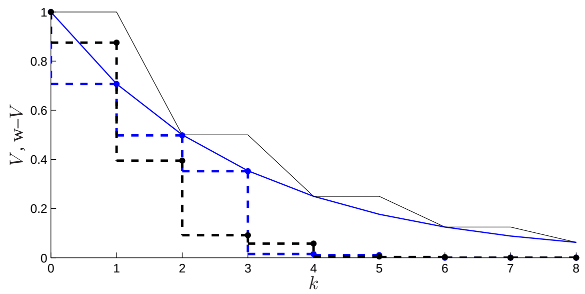

Fig. 2. The Lyapunov functions (V) obtained from the synthesis method presented (dashed black) and from application of the PLL (dashed blue), evaluated along the trajectory of the initial condition $x(0) = [-0.9713 \quad 2.9531]^\top$. The worst–case decrease of the Lyapunov functions (w–V) is shown in black and blue line respectively.

Application of the two methods led to two different PPI sequences of ellipsoids, namely $\{\mathbb{E}(k)\}_{k\in\mathbb{Z}_{[0,1]}}$ for the synthesis method presented in Section 3, and $\{\mathbb{E}_{\text{PLL}}(k)\}_{k\in\mathbb{Z}_{[0,1]}}$ when the PLL Lemma was applied, with corresponding estimates of the ROA of the closed–loop system $\hat{\mathcal{R}}(\mathbb{X}) := \mathbb{E}(0)$ and $\hat{\mathcal{R}}_{\text{PLL}}(\mathbb{X}) := \mathbb{E}_{\text{PLL}}(0)$ respectively. As seen from Figure 1, the estimate of the ROA $\hat{\mathcal{R}}(\mathbb{X})$ is significantly larger than $\hat{\mathcal{R}}_{\text{PLL}}(\mathbb{X})$. It is worth reminding that in both synthesis methods the same objective function is minimized, implying that $\{\mathbb{E}_{\text{PLL}}(k)\}_{k\in\mathbb{Z}_{[0,1]}}$ can be generated from a feasible solution to problem (25), while there does not exist a feasible solution that can generate $\{\mathbb{E}(k)\}_{k\in\mathbb{Z}_{[0,1]}}$ by applying the PLL synthesis approach. In the same figure, the trajectories of the closed–loop system corresponding to the synthesis method presented in this article can be seen, with initial condition $x(0) = [-0.9713 \quad 2.9531]$. It is observed that the state constraints are satisfied at all times. Lastly, in Figure 2, the values of the Lyapunov functions for the two synthesis methods are shown in dashed line, together with the upper bounds on their values at each instant, posed by the Lyapunov conditions (solid lines). It can be seen that the Lyapunov function for the method proposed is not required to decrease at each instant as in the PLL synthesis case, but at each period.

5. CONCLUSIONS

A systematic method for obtaining simultaneously a stabilizing linear periodic state feedback law and an estimation of the region of attraction of the closed–loop system was proposed. The synthesis method concerns the relevant class of linear periodic systems that are subject to state and input constraints. An illustrative example demonstrated the effectiveness of the approach compared to the PLL synthesis method.

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